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Research Article

The Analytical Form of the Dispersion Equation of Elastic Waves in Periodically Inhomogeneous Medium of Different Classes of Crystals

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The investigation of thermoelastic wave propagation in elastic media is bound to have much influence in the fields of material science, geophysics, seismology, and so on. The heat conduction equations and bound equations of motions differ by the difficulty level and presence of many physical and mechanical parameters in them. Therefore thermoelasticity is being extensively studied and developed. In this paper by using analytical matrizant method set of equation of motions in elastic media are reduced to equivalent set of first-order differential equations. Moreover, for given set of equations, the structure of fundamental solutions for the general case has been derived and also dispersion relations are obtained.

1. Introduction

The theory of thermoelasticity deals with the study of mutual interactions of thermal and mechanical fields in elastic bodies [1, 2]. It has vast applications in the various branches of Physics as well as in engineering, like materials engineering, mechanical engineering, nuclear engineering, and so forth. Theory of thermoelasticity is based on assumption that temperature distribution in an elastic object is governed by hyperbolic type parabolic-type partial differential equation as described by Fourier law of heat conduction [3–5]. According to Fourier law any thermal impulse is felt everywhere instantly in an object. Obviously it raised some serious concerns due to its unrealistic point of view. In order to circumvent this problem and to make it realistic a generalized theory of thermoelasticity was proposed which takes into account a finite thermal relaxation time. In this theory the temperature distribution is governed by hyperbolic type equations, which results in heat propagation in solids being considered as wave phenomenon instead of diffusion phenomenon.

In order to investigate the wave propagation in anisotropic inhomogeneous medium a new method of matrizant was developed. This method allows investigation of wave propagation in anisotropic medium with various physical and mechanical properties [6–8].

In 1950 Thompson [9] proposed a matrix method in order to investigate the elastic wave propagation in isotropic stratified media. Haskell also enhanced the method in 1953 [10]. After that major work was carried out by Stroh and others [11, 12]. He analytically investigated the dislocations in anisotropic medium by expressing first-order motion equations using (6×6) matrix. In order to investigate the insulators made up of piezoelectric materials, six-dimensional framework was enhanced to eight-dimensional formalism. Matrix method also paved the way for carrying out numerical simulation in anisotropic media [13, 14]. Various researchers have investigated the ordered structures and layered medium by using the matrix method. In this connection, following papers are of particular interest. Wave propagation has been investigated by matrix algebra method [15, 16], WKB method, and ray method [17–19]. Some investigations have tried to employ matricant, in which infinite product of truncated exponentials of the matrix of system coefficients and also Peano expansion are satisfied [7, 8]. However in case of periodic structure, Peano expansion cannot be fully solved. Therefore development of analytical techniques will open new dimensions to understand wave propagation in periodic structure.

In case of layered and periodic medium the dispersion equations have been obtained and also the matricant structure was formed for nonhomogenous isotropic medium [3]. In [4] the matricant was obtained employing Chebyshev-Gegenbauer's polynomial form, for the case of finite periodic inhomogeneous layer. For such structures, the modified conditions in determining the dispersion relationship having mutual transformation of transverse and longitudinal waves are obtained. In [20, 21] these results have been generalized in case of anisotropic inhomogeneous media.

The applications of matrizant method for nondestructive testing and wave propagation in thermoelastic media are considered [22].

Periodically heterogeneous media have lot of importance from applied and theoretical perspective. Wave propagation in discrete periodic structures has been extensively studied. While in case of continuous medium, layered homogeneous isotropic periodic structures are well studied. However the investigations of wave propagation in more complex periodically heterogeneous medium are carried out using various numerical methods or approximate analytical method. One of them is matrizant method; it was initially developed in late 70s in the Kazakh scientific school of Jakhan Suleimenuly Erzhanov, to investigate the dynamics of inhomogeneous medium.

The method aims at reducing original equations of motion, by using method of separation of variables, to an equivalent system of ordinary differential equations of first order with variable coefficients. After that the resulting system of equations defines the structure of matrizant.

2. Elastic Waves

The motion equations in elastic medium and generalized Hooke's law describe wave propagation in elastic media as follows [21]:

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \qquad \sigma_{ij} = c_{ijkl}\varepsilon_{kl}, \tag{1}$$

where $\varepsilon_{kl} = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ represents the deformation tensor components, u_i denotes the mechanical displacement vector, σ_{ij} are the stress tensor, c_{ijkl} represents the elastic parameters of nonisotropic media, and density of medium is represented by ρ.

The medium is assumed to be stratified; that is, parameters employed to describe the material depend on space variable along z-axis.

Using the representation of the solution

$$f(x, y, z, t) = f(z)\exp\left(i\omega t - ik_x x - ik_y y\right), \tag{2}$$

where ω is radial frequency, k_i are a projection of wave number. The multiplier $\exp(i\omega t - ik_x x - ik_y y)$ is omitted in the following equations for clarity.

Taking into consideration propagation direction, derivative of anisotropic medium on z-axis, and using (2) then (1) are reduced to a system of first-order ODEs having variable coefficients.

$$\frac{d\vec{\mathbf{W}}}{dz} = \mathbf{B}\vec{\mathbf{W}}; \qquad \vec{\mathbf{W}} = \left(u_z, \sigma_{zz}, u_x, \sigma_{xz}, u_y, \sigma_{yz}\right)^t; \tag{3}$$

the transposition operator is denoted by t.

The coefficient matrix $\mathbf{B}$, for the case of triclinic anisotropic medium, takes the following form:

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & 0 & 0 & b_{24} & 0 & b_{26} \\ b_{24} & b_{14} & b_{33} & b_{34} & b_{35} & b_{36} \\ 0 & b_{13} & b_{43} & b_{33} & b_{45} & b_{46} \\ b_{26} & b_{16} & b_{46} & b_{36} & b_{55} & b_{56} \\ 0 & b_{15} & b_{45} & b_{35} & b_{65} & b_{55} \end{pmatrix}. \tag{4}$$

For the orthorhombic anisotropy:

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 & b_{15} & 0 \\ b_{21} & 0 & 0 & b_{24} & 0 & b_{26} \\ b_{24} & 0 & 0 & b_{34} & 0 & b_{36} \\ 0 & b_{13} & b_{43} & 0 & b_{45} & 0 \\ b_{26} & 0 & 0 & b_{36} & 0 & b_{56} \\ 0 & b_{15} & b_{45} & 0 & b_{65} & 0 \end{pmatrix}. \tag{5}$$

If the vector, representing the direction of wave propagation, is in (xz) plane of anisotropy orthorhombic medium, the coefficient matrix given in (5) is divided into 4.

If the wave propagation direction vector lies in the media matrix (5) splits into (4×4) and (2×2) matrices:

$$\mathbf{B}' = \begin{pmatrix} 0 & b_{12} & b_{13} & 0 \\ b_{21} & 0 & 0 & b_{24} \\ b_{24} & 0 & 0 & b_{34} \\ 0 & b_{13} & b_{43} & 0 \end{pmatrix}; \qquad \vec{\mathbf{W}} = \left(u_z, \sigma_{zz}, u_x, \sigma_{xz}\right)^t; \qquad \mathbf{B}'' = \begin{pmatrix} 0 & b_{56} \\ b_{65} & 0 \end{pmatrix}; \qquad \vec{\mathbf{W}} = \left(u_y, \sigma_{yz}\right)^t. \tag{6}$$

3. Matricant Structure and Its Implications

3.1. Matricant Method. In order to describe the wave propagation in elastic medium various analytical approaches are used, like the formalism proposed by Stroh–Barnett–Lothe [23] for piezoelasticity, the state vector "W" method [13]. The matricant approach is different from others analytical techniques used to investigate wave propagation in elastic medium. In the matricant approach the vector W of coefficient matrix B is chosen in (u_i, σ_{ij}), (θ, q_i), pairs, for instance. The selection of pairs depends on the type of wave to be investigated. Coupled waves in the general case and along main crystal axes are suitably described by the use of such notations.

Solutions of (5) are written as

$$\vec{\mathbf{W}}(z) = \mathbf{T}(z, z_0)\,\vec{\mathbf{W}}(z_0) \tag{7}$$

Here, $\mathbf{T}(z, z_0)$ is the matricant, that is, the normalized fundamental solution matrix of the systems of ODEs. For $\mathbf{T}(z, z_0)$ and $\mathbf{T}^{-1}(z, z_0)$, there are representations in the form of the infinite matrix integral series of exponential type as follows [9, 10]:

$$\mathbf{T}(z, z_0) = \mathbf{I} + \int_{z_0}^{z} \mathbf{B}(z_1)\,\mathrm{d}z_1 + \int_{z_0}^{z}\int_{z_0}^{z_1} \mathbf{B}(z_1)\,\mathbf{B}(z_2)\,\mathrm{d}z_1\mathrm{d}z_2 + \cdots \tag{8}$$

$$\mathbf{T}^{-1}(z, z_0) = \mathbf{I} - \int_{0}^{z} \mathbf{B_1}(z_1)\,\mathrm{d}z_1 + \int_{0}^{z}\int_{0}^{z_1} \mathbf{B_2}(z_2)\,\mathbf{B_1}(z_1)\,\mathrm{d}z_1\mathrm{d}z - \cdots; \tag{9}$$

the identity matrix is denoted by $\mathbf{I}$.

The expansion in (9) is the alternating-sign series with reverse argument ordering of the integrated product of $\mathbf{B}(z_i)$. Note that the matrix $\mathbf{B}(z_i)$ does not commute. As the initial system of equations are satisfied by the matricant, so the successive approximation methods can be used to obtain (8).

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}z} = \mathbf{BT}. \tag{10}$$

It follows from substitution of (7) into (10).

Similar to (10), inverse matricant $\mathbf{T}^{-1}$ is the solution of the equation

$$\frac{\mathrm{d}\mathbf{T}^{-1}}{\mathrm{d}z} = -\mathbf{T}^{-1}\mathbf{B}. \tag{11}$$

It follows from differentiation that the identity

$$\mathbf{TT}^{-1} \equiv \mathbf{T}^{-1}\mathbf{T} \equiv \mathbf{I} \tag{12}$$

The solutions of (2×2) order matrix are well known.

$$\mathbf{B} = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}; \tag{13}$$

the matricants have the structure

$$\mathbf{T} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}; \quad \mathbf{T}^{-1} = \begin{pmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{pmatrix}. \tag{14}$$

For unimodular matrices the above result can be carried out.

In case of $(4{\times}4)$ coefficient matrix in (6), its inverse matrix is given by

$$\mathbf{T}^{-1} = \begin{bmatrix} t_{22} & -t_{12} & -t_{42} & t_{32} \\ -t_{21} & t_{11} & t_{41} & -t_{31} \\ -t_{24} & t_{14} & t_{44} & -t_{34} \\ t_{23} & -t_{13} & -t_{43} & t_{33} \end{bmatrix}. \tag{15}$$

The result given in (15) defines the properties of solutions of systems of first-order ODEs with variable coefficients. This has been obtained with term to term comparison of elements of (8) and (9) and with the help of mathematical induction [5, 24].

In the case of the coefficient matrix (5), the (6×6) matricant structure is obtained in the following form:

$$\mathbf{T}^{-1} = \begin{bmatrix} t_{22} & -t_{12} & -t_{42} & t_{32} & -t_{62} & t_{52} \\ -t_{21} & t_{11} & t_{41} & -t_{31} & t_{61} & -t_{51} \\ -t_{24} & t_{14} & t_{44} & -t_{34} & t_{64} & -t_{54} \\ t_{23} & -t_{13} & -t_{43} & t_{33} & -t_{63} & t_{53} \\ -t_{26} & t_{16} & t_{46} & -t_{36} & t_{66} & -t_{56} \\ t_{25} & -t_{15} & -t_{45} & t_{35} & -t_{65} & t_{55} \end{bmatrix}. \tag{16}$$

It may be noted that elements of $\mathbf{T}^{-1}$ contain only of elements from matricant $\mathbf{T}$. It is a one to one correspondence of elements of direct and its inverse matricant given in (15) and (16). The conservation laws are contained by invariant relationships; these laws have to be satisfied in wave processes. In 1D inhomogeneous isotropic medium having various crystals symmetry is described by matricant structure. The matricant of coefficient matrix is of order $(2n \times 2n)$ [5].

3.2. Periodic Structures. Suppose the variation in parameters that describes medium is $f_i(z)$. The condition $f_i(z + h) = f_i(z)$ is satisfied by periodic structure and also by parameters altered by environment. The period of inhomogeneity is denoted by h. Monodromy matrix is matricant of single period of inhomogeneity. It is known for single period inhomogeneity that $\vec{\mathbf{W}}(h) = \mathbf{T}(0, h)\vec{\mathbf{W}}_{\mathbf{0}}(0)$ and the Floquet–Bloch conditions $\vec{\mathbf{W}}(z + h) = \vec{\mathbf{W}}_{\mathbf{0}}(z)\exp(ik_z h)$. From this follows the following equality:

$$(\mathbf{T} - \mathbf{I}\exp(ik_z h))\,\vec{\mathbf{W}}_{\mathbf{0}} = 0; \tag{17}$$

the equivalent condition are obtained by multiplying (17) with $\mathbf{T}^{-1}e^{ikh}$.

$$\left(\mathbf{T}^{-1} - \mathbf{I}\exp\left(-ik_z h\right)\right)\vec{\mathbf{W}}_{\mathbf{0}} = 0. \tag{18}$$

Using auxiliary matrix and (17) and (18), the modified conditions as given below are derived as follows:

$$\left(\widehat{\mathbf{p}} - \mathbf{I}\cos k_z h\right)\vec{\mathbf{W}}_{\mathbf{0}} = 0, \qquad \widehat{\mathbf{p}} = \frac{1}{2}\left(\mathbf{T} + \mathbf{T}^{-1}\right). \tag{19}$$

Matrix $\widehat{\mathbf{p}}$ for cases (6) is given by

$$\widehat{\mathbf{p}} = \begin{pmatrix} p_{11} & 0 & p_{13} & p_{14} \\ 0 & p_{11} & p_{23} & p_{24} \\ -p_{24} & p_{14} & p_{22} & 0 \\ p_{23} & -p_{13} & 0 & p_{22} \end{pmatrix}. \tag{20}$$

The dispersion in periodic structure is determined by characteristics of (19).

$$s\det\left|\widehat{\mathbf{p}} - \mathbf{I}\cos k_z h\right| = 0. \tag{21}$$

The dispersion equation in case of (4×4) matrices is found by

$$\left.\begin{aligned}\cos k_{1z}h &= \widetilde{p}_1 \\ \cos k_{2z}h &= \widetilde{p}_2\end{aligned}\right\} = \frac{1}{2}\left(p_{11} + p_{22}\right) \pm \frac{1}{2}\sqrt{\left(p_{11} - p_{22}\right)^2 + 4\left(p_{14}p_{23} - p_{13}p_{24}\right)}. \tag{22}$$

The equation of dispersion is obtained. However, the order of characteristic equation is reduced by half when the conditions as laid down in (19) are imposed.

Following recurrence relationship can be obtained from (19) as follows:

$$\mathbf{T}^2 = 2\widehat{\mathbf{p}}\mathbf{T} - \mathbf{I}. \tag{23}$$

Matricant representing periodically inhomogeneous layers can be obtained applying (24).

In the presence of n periods of heterogeneity in form $H = nh$, we can obtain

$$\mathbf{T}(H) = \mathbf{T}_{\mathbf{m}}^{\mathbf{n}}(h) = \mathbf{P}_{\mathbf{n}}(\widehat{\mathbf{p}})\,\mathbf{T}_{\mathbf{m}} - \mathbf{P}_{\mathbf{n-1}}(\widehat{\mathbf{p}}), \tag{24}$$

where $\mathbf{T}_{\mathbf{m}} = \mathbf{T}(h)$ represents the monodromy matrix and $\mathbf{P}_{\mathbf{n}}(\widehat{\mathbf{p}})$ denotes matrix polynomials of Chebyshev-Gegenbauer [5, 25]. The results of Brillouin and Parodi [26] are generalized by above equations.

3.3. Structuring the Matrizant. Structuring the matrizant (normalized matrix of fundamental solutions) is based on its representation in the form of the exponential matrix series [5, 7], (8) and (9).

These matrix series converge absolutely and uniformly on any finite interval. In this case, the following relation is true, (12).

For the matrizant the following expressions also hold:

(i) $T(z_0, z) = T(z_1, z)T(z_0, z_1)$.

(ii) $\ln|T(z_0, z)| = \int_{z_0}^{z} spB(z_1)dz_1$.

(iii) If $B = B_0$ – constant matrix, then $T_0 = \exp[B_0(z - z_0)]$.

(iv) $dT^{-1}/dz = -T^{-1}B$.

Matrix series (8) and (9) can be written in summation form

$$T = \sum_{n=0}^{\infty} T_{(n)}, \qquad T^{-1} = \sum_{n=0}^{\infty} T_{(n)}^{-1}. \tag{25}$$

The index n corresponds to the number of multiplied under signs of integral matrices, where $B(z_i)$ is the number of integrals of the matrix in each term of the series as given in (1). Moreover, the terms of the series with even and odd values of n are as follows:

$$\begin{aligned} T_{\text{even}} &= \sum_{n=0}^{\infty} T_{(2n)}, \\ T_{\text{odd}} &= \sum_{n=0}^{\infty} T_{(2n+1)} \\ T_{\text{even}}^{-1} &= \sum_{n=0}^{\infty} T_{(2n)}^{-1}, \\ T_{\text{odd}}^{-1} &= \sum_{n=0}^{\infty} T_{(2n+1)}^{-1}, \end{aligned} \qquad (n = 0, 1, 2, 3, \ldots). \tag{26}$$

In such case constructing the matrizant is basically expressing the relationships between the elements of the T and T^{-1} matrices, it is based on the element-wise comparison.

As a first approximation $t_{ij}^{(1)} = -t_{ij}^{(1)}$, $t_{ij}^{(-1)} = \int_0^z b_{ij}(z)dz$.

Elastic waves propagating in the orthorhombic syngony of the classes mm2 and 222, hexagonal syngony (6, $\overline{6}$, 622, 6mm, $\overline{6}$m2), tetragonal syngony (class 422), and matrizant structure are built based on the structure of the coefficient matrix, based on the element-wise comparison of matrix T and T^{-1}. The structure of the coefficient matrix in the bulk case is as follows [5]:

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & 0 & b_{15} & 0 \\ b_{21} & 0 & 0 & b_{24} & 0 & b_{26} \\ b_{24} & 0 & 0 & b_{34} & 0 & 0 \\ 0 & b_{13} & b_{43} & 0 & b_{45} & 0 \\ b_{26} & 0 & 0 & 0 & 0 & b_{56} \\ 0 & b_{15} & b_{45} & 0 & b_{65} & 0 \end{bmatrix}. \tag{27}$$

See (3).

$$\vec{\mathbf{W}} = \left(u_z, \sigma_{zz}, u_x, \sigma_{xz}, u_y, \sigma_{yz}\right)^t. \tag{28}$$

The coefficients of the matrix b_{ij} (27) for the crystals of orthorhombic syngony are equal to

$$\begin{aligned}
b_{12} &= \frac{1}{c_{33}};\\
b_{13} &= b_{42} = ik_x \frac{c_{13}}{c_{33}};\\
b_{15} &= b_{62} = ik_y \frac{c_{32}}{c_{33}};\\
b_{21} &= -\rho\omega^2;\\
b_{24} &= b_{31} = ik_x;\\
b_{26} &= b_{51} = ik_y;\\
b_{34} &= \frac{1}{c_{55}};\\
b_{43} &= k_y^2 c_{66} - \rho\omega^2 + k_x^2\left(c_{11} - \frac{c_{13}^2}{c_{33}}\right);\\
b_{45} &= b_{63} = \left(c_{66} + c_{12} - \frac{c_{13}c_{32}}{c_{33}}\right)k_x k_y;\\
b_{56} &= \frac{1}{c_{44}};\\
b_{65} &= k_x^2 c_{66} - \rho\omega^2 + \left(c_{22} - \frac{c_{23}^2}{c_{33}}\right)k_y^2,
\end{aligned} \tag{29}$$

where $c_{\alpha\beta}$- are elastic parameters, k_x, k_y- are components of the wave vector, ρ is medium density, and ω isangular frequency.

Coefficients b_{ij} for hexagonal syngony (classes 6, $\overline{6}$, 622, 6mm, $\overline{6}$m2) have the form

$$\begin{aligned}
b_{12} &= \frac{1}{c_{33}};\\
b_{13} &= ik_x \frac{c_{13}}{c_{33}};\\
b_{15} &= ik_y \frac{c_{13}}{c_{33}};\\
b_{21} &= -\rho\omega^2;\\
b_{24} &= ik_x;\\
b_{26} &= ik_y;\\
b_{34} &= b_{56} = \frac{1}{c_{44}};\\
b_{43} &= k_y^2\left(\frac{c_{11}-c_{12}}{2}\right) + k_x^2\left(c_{11} - \frac{c_{13}^2}{c_{33}}\right) - \rho\omega^2;\\
b_{45} &= \left(c_{12} + \frac{c_{11}-c_{12}}{2} - \frac{c_{13}^2}{c_{33}}\right)k_x k_y;\\
b_{65} &= k_x^2\frac{c_{11}-c_{12}}{2} + \left(c_{11} - \frac{c_{13}^2}{c_{33}}\right)k_y^2 - \rho\omega^2.
\end{aligned} \tag{30}$$

Coefficients b_{ij} for tetragonal syngony (class 422) have the form

$$\begin{aligned}
b_{12} &= \frac{1}{c_{33}};\\
b_{13} &= ik_x \frac{c_{13}}{c_{33}};\\
b_{15} &= ik_y \frac{c_{13}}{c_{33}};\\
b_{21} &= -\rho\omega^2;\\
b_{24} &= ik_x;\\
b_{26} &= ik_y;\\
b_{34} &= b_{56} = \frac{1}{c_{44}};\\
b_{43} &= k_y^2 c_{66} + k_x^2\left(c_{11} - \frac{c_{13}^2}{c_{33}}\right) - \rho\omega^2;\\
b_{45} &= \left(c_{12} + c_{66} - \frac{c_{13}^2}{c_{33}}\right)k_x k_y;\\
b_{65} &= k_x^2 c_{66} + \left(-\frac{c_{13}^2}{c_{33}}\right)k_y^2 - \rho\omega^2.
\end{aligned} \tag{31}$$

As it can be seen from the last relations coefficients b_{ij} of crystals of high and average symmetry differ only in the values of the elastic constants $c_{\alpha\beta}$.

Matrix of order 6 describes the propagation of bound elastic one longitudinal and two transverse waves.

The second approximation matrizant has the form

$$T_{(2)} = \int_0^z \int_0^{z_1} B(z_1) B(z_2)\, dz_1 dz_2. \tag{32}$$

Inverse matrizant in the second approximation takes the form

$$T_{(2)}^{-1} = \int_0^z \int_0^{z_1} B(z_2) B(z_1)\, dz_1 dz_2. \tag{33}$$

Comparison of the terms of the second approximation gives the following relationship between the elements of matrizant T and T^{-1}:

$$T^{-1}_{(2)} = \begin{bmatrix} t_{22} & 0 & 0 & t_{32} & 0 & t_{52} \\ 0 & t_{11} & t_{41} & 0 & t_{61} & 0 \\ 0 & t_{14} & t_{44} & 0 & t_{64} & 0 \\ t_{23} & 0 & 0 & t_{33} & 0 & t_{53} \\ 0 & t_{16} & t_{46} & 0 & t_{66} & 0 \\ t_{25} & 0 & 0 & t_{35} & 0 & t_{55} \end{bmatrix}_{(2)}. \quad (34)$$

The elements t_{ij} are the elements of the direct matrizant (27).

Similarly, elements of the third approximation are compared.

$$T_{(3)} = \iiint B(z_1) B(z_2) B(z_3) \, dz_1 dz_2 dz_3$$

$$\text{or } T_{(3)} = T_{(2)} B(z_3)$$

$$T_{(3)} = \begin{bmatrix} 0 & t_{12} & t_{13} & 0 & t_{15} & 0 \\ t_{21} & 0 & 0 & t_{24} & 0 & t_{26} \\ t_{31} & 0 & 0 & t_{34} & 0 & t_{36} \\ 0 & t_{42} & t_{43} & 0 & t_{45} & 0 \\ t_{51} & 0 & 0 & t_{54} & 0 & t_{56} \\ 0 & t_{62} & t_{63} & 0 & t_{65} & 0 \end{bmatrix}_{(3)}. \quad (35)$$

Inverse matrizant in the third approximation has the form

$$T^{-1}_{(3)} = \iiint B(z_3) B(z_2) B(z_1) \, dz_1 dz_2 dz_3$$
$$\text{or } T^{-1}_{(3)} = B(z_3) T^{-1}{}_{(2)} \quad (36)$$

and has the structure

$$T^{-1}_{(3)} = \begin{bmatrix} 0 & t_{12} & t_{42} & 0 & t_{62} & 0 \\ t_{21} & 0 & 0 & t_{15} & 0 & t_{51} \\ t_{24} & 0 & 0 & t_{34} & 0 & t_{54} \\ 0 & t_{13} & t_{43} & 0 & t_{63} & 0 \\ t_{26} & 0 & 0 & t_{36} & 0 & t_{56} \\ 0 & t_{15} & t_{45} & 0 & t_{65} & 0 \end{bmatrix}_{(3)}. \quad (37)$$

The elements t_{ij} are the elements of the matrizant (35). Endless rows of the matrix can be written as follows [5]:

$$T = T_{\text{even}} + T_{\text{odd}},$$
$$T^{-1} = T^{-1}_{\text{even}} - T^{-1}_{\text{odd}}, \quad (38)$$

where $T^{\pm 1}$ – corresponds to the sum of odd and even rows (9, 10).

Mathematical induction proves that the structure of the $T^{-1}_{(n)}$ is preserved for any n.

Structure (34) is valid for all even T^{-1} and the structure of (37) is valid for all odd values of T^{-1}. Generalizing (34) and (37) according to (38), we obtain the structure of the T^{-1} matrizant as follows [5]:

$$T^{-1} = \begin{bmatrix} t_{22} & -t_{12} & -t_{42} & t_{32} & -t_{62} & t_{52} \\ -t_{21} & t_{11} & t_{41} & -t_{31} & t_{61} & -t_{51} \\ -t_{24} & t_{14} & t_{44} & -t_{34} & t_{64} & -t_{54} \\ t_{23} & -t_{13} & -t_{43} & t_{33} & -t_{63} & t_{53} \\ -t_{26} & t_{16} & t_{46} & -t_{36} & t_{66} & -t_{56} \\ t_{25} & -t_{15} & -t_{45} & t_{35} & -t_{65} & t_{55} \end{bmatrix}. \quad (39)$$

Elements t_{ij} of matrices T^{-1}_{even} and T^{-1}_{odd} are elements of matrices; T_{even} and T_{odd} are the elements of direct matrix T, respectively.

Thus, the structure of matrizant is a relationship between elements of the forward and inverse matrizant in the form (39), as well as the relationship between the elements T and T^{-1} as follows from (12).

Analytical representation of matrizant of periodically inhomogeneous layer is derived from knowing the structure T^{-1}.

4. Dispersion Equations for the Elastic Anisotropic Mediums

We introduce the following matrix [5]:

$$\hat{p} = \frac{1}{2}\left(T + T^{-1}\right); \quad (40)$$

substituting values of T and T^{-1} in (40), we obtain

$$2\hat{p} = \begin{pmatrix} p_1 & 0 & p_{13} & p_{14} & p_{15} & p_{16} \\ 0 & p_1 & p_{23} & p_{24} & p_{25} & p_{26} \\ -p_{24} & p_{14} & p_3 & 0 & p_{35} & p_{36} \\ p_{23} & -p_{13} & 0 & p_3 & p_{45} & p_{46} \\ -p_{26} & p_{16} & p_{46} & -p_{36} & p_5 & 0 \\ p_{25} & -p_{15} & -p_{45} & p_{35} & 0 & p_5 \end{pmatrix}. \quad (41)$$

The matrix p as given in (40) which is important for the regular structures gives the recurrence relation as in (23).

Consistent application of (23) gives possibility of representing T^n in the form

$$T^n = P_n(p) T - P_{n-1}(p), \quad (42)$$

where $P_n(p)$ is Chebyshev-Gegenbaur matrix polynomials of the second kind.

Equation (42) allows obtaining T^n in an explicit analytic form

$$T^n = \sum_{i=1}^{3} P_i \left[P_n(\tilde{p}_i) T - P_n(\tilde{p}_i) \mathbf{I} \right], \quad (43)$$

where $P_i = (1/(\tilde{p}_i - \tilde{p}_j)(\tilde{p}_i - \tilde{p}_k))[\hat{p} - \tilde{p}_j\mathbf{I}][\hat{p} - \tilde{p}_k\mathbf{I}]$, i, j, k-$1, 2, 3$; $i \neq j$, $j \neq k$, $i \neq k$.

$\tilde{p}_1, \tilde{p}_{21}, \tilde{p}_{31}$ are the roots of the characteristic equation, satisfying the following condition (23).

The matrix method of matrizant allows twice lowering the degree of the characteristic equation, which in the end has the form

$$\left[\lambda^3 + a\lambda^2 + b\lambda + c\right]^2 = 0$$
$$\lambda^3 + a\lambda^2 + b\lambda + c = 0, \qquad (44)$$

where

$$\begin{aligned}
\lambda &= \cos \tilde{k} h_i \\
a &= -\left(p_1 + p_2 + p_5\right) \\
b &= p_1 p_3 + p_1 p_5 + p_3 p_5 - p_{14} p_{23} + p_{13} p_{24} - p_{16} p_{25} \\
&\quad + p_{15} p_{26} + p_{35} p_{46} \\
c &= -p_1\left(p_3 p_5 + p_{36} p_{45} - p_{35} p_{46}\right) \\
&\quad + p_3\left(p_{16} p_{25} - p_{15} p_{26}\right) + p_5\left(p_{14} p_{23} - p_{13} p_{24}\right) - \\
&\quad - p_{16}\left(p_{23} p_{35} + p_{24} p_{45}\right) + p_{13}\left(p_{26} p_{35} - p_{25} p_{36}\right) \\
&\quad + p_{15}\left(p_{23} p_{36} + p_{24} p_{46}\right) + p_{14}\left(p_{26} p_{45} - p_{25} p_{46}\right).
\end{aligned} \qquad (45)$$

The solution of the characteristic equation (44) gives three roots, which have the following form:

$$\begin{aligned}
\cos \tilde{k} h_1 &= \frac{1}{6}\left(1.58 \sqrt[3]{\delta} - 2a + \frac{2.52\left(a^2 - 3b\right)}{\sqrt[3]{\delta}}\right) \\
\cos \tilde{k} h_2 &= \frac{1}{12}\left(-3.17 \sqrt[3]{\delta} - 4a - \frac{4\sqrt[3]{-2}\left(a^2 - 3b\right)}{\sqrt[3]{\delta}}\right) \\
\cos \tilde{k} h_3 &= \frac{1}{12}\left(-3.17 \sqrt[3]{-1} \sqrt[3]{\delta} - 4a + \frac{5\left(a^2 - 3b\right)}{\sqrt[3]{\delta}}\right),
\end{aligned} \qquad (46)$$

where $\delta = -2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3}$.

Relations (46) determine the dispersion equations of elastic waves in above-mentioned crystals. The difference lies in the roots of the difference between the coefficients b_{ij} in the matrix (27).

5. Conclusion

In this paper we have developed the structure of matrizant and from it obtained invariant relations which reflects the inner symmetry of inner equations and contains conservation laws. Also we have derived an analytical representation of matrizant of periodically inhomogeneous layer using Chebyshev-Gegenbauer polynomials and obtained a separate dispersion equation. Finally, by using different crystals systems the analytical solution of equations of motion for a wide class of homogenous anisotropic medium has been obtained.

Competing Interests

The authors declare that they have no competing interests.

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