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PAPER

Propagation of electromagnetic waves in stationary anisotropic media

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Abstract

In this paper we have investigated the fundamental properties of the solutions of Maxwell's equations describing free electromagnetic waves in stationary anisotropic media, with tensor characteristics along the Z axis. A matrix of coefficients, the structure of the matrix of Maxwell's equations, dispersion equations and indicatrix curves for a transparent anisotropic one-dimensional inhomogeneous conductive medium have been obtained. Exact analytical solutions in case of homogeneous anisotropic media are constructed on the basis of the matrix structure.

1. Introduction

The propagation of electromagnetic and elastic waves in media has vast applications in the area of advanced device design. There is a connection between the deformation of medium and production of electromagnetic field, like piezoelectric, piezomagnetic, magnetostrictive and thermo piezoelectric materials. The electromagnetic waves propagation in such materials is of much concern and has been investigated by many researchers. In reference [1] formulation has been developed to investigate the electromagnetic waves scattering in an isotropic media by employing transfer matrix. It is shown that diffraction of electromagnetic waves due to half plane in an anisotropic medium reduces to the correct field expressions for limiting cases [2]. In [3] EM waves propagation through the medium boundaries has been investigated by developing analytical algorithm. The refraction and reflection of EM waves in an anisotropic medium reveals that there is a case in which pointing vector is parallel to the wave [4]. Moreover, in case of quasi-homogeneous medium the propagation of EM waves reveal that spectral density and the spectral degree of coherence of the scattered field can be factorized as a product of two parts, the one is dependent on the polarization of the incident field, and the other is dependent on the characteristics of the medium [5]. The variation in EM wave polarization state in dissipative anisotropic media has been analyzed by using in quasi-isotropic approximation and differential equations so obtained are written in terms of a dielectric tensor of birefringent media with dissipation [6].

Numerical study on EM wave propagation in case of planar waveguides with chirality reveals the phenomenon of backward waves [7]. TM modes of scattered EM waves have been studies by using Factorization method which gave simple criterion to compute a picture of shape of diffraction gratings in a rapid way [8]. In [9], the linear coupling of EM waves is considered to be a indication of the polarization degeneration of the Maxwell equations in a weakly inhomogeneous, non-1D medium. It has been revealed that the presence of two polarization-degenerate normal waves imposes strong constraints on dielectric tensor components near the interaction region. As a result, the possible types of linear wave coupling and their associated wave equations recognize a universal classification that is independent of linear medium modeling. Moreover, in case of anisotropic dielectric medium with two generic matrices epsilon (ij) and mu (ij) of permittivity and permeability EM wave propagation gives a compact tensorial dispersion relation. The main advantage of such derivation is that it does not need the matrices must be positive, invertible or symmetric [10]. In uniaxial materials with epsilon(1) = epsilon(2), mu(1) = mu(2), analytical expressions are obtained for the components of two harmonic plane waves propagating with different refractive indexes [11]. Based on the method matriciant [12] EM wave propagation in elastic, thermoelastic anisotropic, anisotropic dielectric media, in anisotropic plates, and with magnetoelectric effect have been investigated [13–16]. The matriciant method is also employed to investigate propagation of spin waves, the waves in liquid crystals and wave propagation in thermoelastic media [17–19].

The construction of the matriciant structure allows us to generalize the classical results of Brillouin and Parodi for discrete periodic structures to the case of continuously inhomogeneous anisotropic continuous media.

In this paper, we present the results of the problem of the propagation of electromagnetic waves in one-dimensional inhomogeneous anisotropic media, and obtain the indicatrix curves, phase and group velocities. Moreover, a unified description of electro-magnetic waves in piezoelectric, piezomagnetic and magnetoelectric media is also obtained; an analytical solution to the problems of reflection/refraction at the boundary of anisotropic media, including in the presence of coupled fields, is obtained. The equations of the wave velocity indicatris are determined.

2. The research method

Matriciant—the fundamental matrix $X(t)$ of solutions to a system of ordinary differential equations $x'(t) = A(t)x(t)$, $x(t) \in R^n$, $A(t)$—is a one-parameter family of matrices normalized at t_0 [20, 21].

The research method is the matricant method [12], which allows obtaining accurate analytical solutions of differential equations describing the electromagnetic wave propagation materials with piezoelectric, piezomagnetic and thermo-piezoelectric properties.

The essence of the method is to reduce the initial equations, based on the method of separation of variables (representing the solution in the form of plane waves), to an equivalent system of ordinary differential equations of the first order with variable coefficients. For the resulting system of equations, the matriciant structure is determined (normalized matrix of fundamental solutions).

The matrix method has been tested and the results obtained are consistent with previously known phenomenon [22].

3. Basic equations and formulation

Electromagnetic waves in material media are described by Maxwell's equations [23–26]:

$$rot\vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad rot\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}, \quad div\vec{D} = \rho, \quad div\vec{B} = 0 \tag{1}$$

in which ρ and $\vec{j}$ represents volume charge density and current density vector respectively.

The harmonics of wave propagation in various anisotropic media are determined by the structure of tensors $\hat{\varepsilon}$, $\hat{\mu}$, $\hat{\sigma}$ and by the dependence of the components of these tensors on the frequency and wave vector using dispersion equations):

$$\varepsilon_{ij} = \varepsilon_{ij}(\omega, \vec{k}),\ \mu_{ij} = \mu_{ij}(\omega, \vec{k}),\ \sigma_{ij} = \sigma_{ij}(\omega, \vec{k}) \quad (\mathrm{i, j} = 1, 2, 3, \text{ or } \mathrm{i, j} = \mathrm{x, y, z}).$$

In this case, the functions $\varepsilon_{ij} = \varepsilon_{ij}(z)$, $\mu_{ij} = \mu_{ij}(z)$, $\sigma_{ij} = \sigma_{ij}(z)$ are generally assumed to be piecewise continuous.

Considering conducting media, we will take into account induced currents only. The volumetric charge density is assumed to be zero ($\rho = 0$). The tensor characteristics will be assumed to be symmetric, we get:

$$\varepsilon_{ij} = \varepsilon_{ji}, \quad \mu_{ij} = \mu_{ji}, \quad \sigma_{ij} = \sigma_{ji}. \tag{2}$$

The problem posed in Anisotropic media is the presence of abundance of parameters. One of the constructive ways to overcome these difficulties is a consistent and detailed study of the properties of solutions to Maxwell's equations. To establish the laws of these solutions from the structure of tensor quantities that determines the anisotropy of the medium. Such a study has to be carried out on the basis of the simplest possible waves of a fairly general nature.

This study examines the time-harmonic solutions of Maxwell's equations and the method of separating variables with respect to spatial coordinates.

Numbering coordinates (x, y, z) with numbers 1, 2, 3, the relationship between the vectors of inductions and intensities of the electromagnetic field and $\vec{D}$ и $\vec{E}$, $\vec{B}$ and $\vec{H}$ can be represented as:

$$D_i = \varepsilon_0 \varepsilon_{ij} E_j$$
$$B_i = \mu_0 \mu_{ij} H_j \tag{3}$$

In conducting media, these material equations are supplemented by the following:

$$j_i = \sigma_{ij} E_j \tag{4}$$

Taking into account the above initial positions, the representation of solutions of wave fields $\vec{E}$, $\vec{H}$, $\vec{B}$, $\vec{D}$ is considered in the following:

$$\vec{f} = \vec{f}(\omega, z) e^{i(\omega t \pm k_x x \pm k_y y)}, \tag{5}$$

Where ω is cyclic frequency and k_x, k_y represents the x and y components of the wave vector $\vec{k}$. The properties of the medium do not depend on the coordinates x and y, i.e. the medium is assumed to be inhomogeneous along the Oz axis.

Taking the divergence from the left sides of the first pair of equation (1), we get:

$$div rot \vec{E} = 0, \quad div rot \vec{H} = 0,$$

which gives

$$div \vec{B} = 0, \quad div \vec{D} = 0.$$

Then the original system of equations (1)–(5) becomes as follows:

$$rot_i \vec{E} = -i\omega\mu_0 \mu_{ij} H_j, \quad rot_i \vec{H} = i(\omega\varepsilon_0\varepsilon_{ij} - i\sigma_{ij}) E_j. \tag{6}$$

These equations are written in component wise form as follows:

$$\begin{aligned}
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -i\omega\mu_0(\mu_x H_x + \mu_{xy} H_y + \mu_{xz} H_z);\\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -i\omega\mu_0(\mu_{yx} H_x + \mu_y H_y + \mu_{yz} H_z);\\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -i\omega\mu_0(\mu_{zx} H_x + \mu_{zy} H_y + \mu_z H_z);\\
\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= i(\omega\varepsilon_0\varepsilon_x - i\sigma_x)E_x + i(\omega\varepsilon_0\varepsilon_{xy} - i\sigma_{xy})E_y + i(\omega\varepsilon_0\varepsilon_{xz} - i\sigma_{xz})E_z;\\
\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= i(\omega\varepsilon_0\varepsilon_{yx} - i\sigma_{yx})E_x + i(\omega\varepsilon_0\varepsilon_y - i\sigma_y)E_y + i(\omega\varepsilon_0\varepsilon_{yz} - i\sigma_{yz})E_z;\\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= i(\omega\varepsilon_0\varepsilon_{zx} - i\sigma_{zx})E_x + i(\omega\varepsilon_0\varepsilon_{zy} - i\sigma_{zy})E_y + i(\omega\varepsilon_0\varepsilon_z - i\sigma_z)E_z.
\end{aligned} \tag{7}$$

Using the representation of solutions for the vectors $\vec{E}$ and $\vec{H}$ in the form (5):

$$\begin{aligned}
E_i(x, y, z, t) &= E_i(z, \omega) e^{i(\omega t - k_x x - k_y y)},\\
H_i(x, y, z, t) &= H_i(z, \omega) e^{i(\omega t - k_x x - k_y y)},
\end{aligned} \tag{8}$$

and, substituting (8) into (7), we obtain a system of four differential and two algebraic equations:

$$\begin{aligned}
\frac{dE_y}{dz} &= -ik_y E_z + i\omega\mu_0(\mu_x H_x + \mu_{xy} H_y + \mu_{xz} H_z);\\
\frac{dH_x}{dz} &= -ik_x H_z + i(\omega\varepsilon_0\varepsilon_{yx} - i\sigma_{yx})E_x + i(\omega\varepsilon_0\varepsilon_y - i\sigma_y)E_y + i(\omega\varepsilon_0\varepsilon_{yz} - i\sigma_{yz})E_z;\\
\frac{dH_y}{dz} &= -ik_y H_z - i(\omega\varepsilon_0\varepsilon_x - i\sigma_x)E_x - i(\omega\varepsilon_0\varepsilon_{xy} - i\sigma_{xy})E_y - i(\omega\varepsilon_0\varepsilon_{xz} - i\sigma_{xz})E_z;\\
\frac{dE_x}{dz} &= -ik_x E_z - i\omega\mu_0(\mu_{yx} H_x + \mu_y H_y + \mu_{yz} H_z);\\
E_z &= -\frac{\omega\varepsilon_0\varepsilon_{zy} - i\sigma_{zy}}{\omega\varepsilon_0\varepsilon_z - i\sigma_z}E_y + \frac{k_y}{\omega\varepsilon_0\varepsilon_z - i\sigma_z}H_x - \frac{k_x}{\omega\varepsilon_0\varepsilon_z - i\sigma_z}H_y - \frac{\omega\varepsilon_0\varepsilon_{zx} - i\sigma_{zx}}{\omega\varepsilon_0\varepsilon_z - i\sigma_z}E_x;\\
H_z &= \frac{k_x}{\omega\mu_0\mu_z}E_y - \frac{\mu_{zx}}{\mu_z}H_x - \frac{\mu_{zy}}{\mu_z}H_y - \frac{k_y}{\omega\mu_0\mu_z}E_x.
\end{aligned} \tag{9}$$

Excluding the quantities E_z and H_z from the system of equation (9), we obtain a closed system of equations regarding E_y, H_x, H_y, E_x (the choice of such an order in the arrangement of the field components is associated with the separation of waves into TE- and TM-polarized ones and their mutual transformation):

$$
\begin{aligned}
\frac{dE_y}{dz} &= i\left(k_y\frac{\omega\varepsilon_0\varepsilon_{zy}-i\sigma_{zy}}{\omega\varepsilon_0\varepsilon_z-i\sigma_z}+k_x\frac{\mu_{xz}}{\mu_{zz}}\right)E_y+i\left(\mu_0\mu_x\omega-\frac{k^2{}_y}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}-\omega\frac{\mu_0\mu_{xz}^2}{\mu_{zz}}\right)H_x\\
&\quad+i\left(\omega\mu_0\mu_{xy}+\frac{k_xk_y}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}-\omega\frac{\mu_0\mu_{xz}\mu_{yz}}{\mu_{zz}}\right)H_y\\
&\quad+i\left(k_y\frac{\omega\varepsilon_0\varepsilon_{zx}-i\sigma_{zx}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}-k_y\frac{\mu_{xz}}{\mu_{zz}}\right)E_x\\
\frac{dH_x}{dz} &= i\left[\omega\varepsilon_0\varepsilon_{yy}-\frac{k^2{}_x}{\omega\mu_0\mu_{zz}}-i\sigma_{yy}-(\omega\varepsilon_0\varepsilon_{yz}-i\sigma_{yz})\frac{\omega\varepsilon_0\varepsilon_{zy}-i\sigma_{zy}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right]E_y\\
&\quad+i\left(k_y\frac{\omega\varepsilon_0\varepsilon_{zy}-i\sigma_{zy}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}+k_x\frac{\mu_{xz}}{\mu_{zz}}\right)H_x+i\left(k_x\frac{\mu_{yz}}{\mu_{zz}}-k_x\frac{\omega\varepsilon_0\varepsilon_{yz}-i\sigma_{yz}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right)H_y\\
&\quad+i\left[\omega\varepsilon_0\varepsilon_{yx}+\frac{k_xk_y}{\omega\mu_0\mu_{zz}}-i\sigma_{yx}-(\omega\varepsilon_0\varepsilon_{yz}-i\sigma_{yz})\frac{\omega\varepsilon_0\varepsilon_{zx}-i\sigma_{zx}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right]E_x\\
\frac{dH_y}{dz} &= -i\left[\omega\varepsilon_0\varepsilon_{yx}+\frac{k_xk_y}{\omega\mu_0\mu_{zz}}-i\sigma_{yx}-(\omega\varepsilon_0\varepsilon_{yz}-i\sigma_{yz})\frac{\omega\varepsilon_0\varepsilon_{zx}-i\sigma_{zx}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right]E_y\\
&\quad-i\left(k_y\frac{\omega\varepsilon_0\varepsilon_{zx}-i\sigma_{zx}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}-k_y\frac{\mu_{xz}}{\mu_{zz}}\right)H_x+i\left(k_y\frac{\mu_{yz}}{\mu_{zz}}+k_x\frac{\omega\varepsilon_0\varepsilon_{xz}-i\sigma_{xz}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right)H_y\\
&\quad+i\left(\frac{k_y^2}{\omega\mu_0\mu_z}-\omega\varepsilon_0\varepsilon_{xx}+i\sigma_{xx}+(\omega\varepsilon_0\varepsilon_{xz}-i\sigma_{xz})\frac{\omega\varepsilon_0\varepsilon_{zx}-i\sigma_{zx}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right)E_x\\
\frac{dE_x}{dz} &= i\left(k_x\frac{\mu_{yz}}{\mu_{zz}}-k_x\frac{\omega\varepsilon_0\varepsilon_{yz}-i\sigma_{yz}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right)E_y-i\left(\omega\mu_0\mu_{xy}+\frac{k_xk_y}{\omega\varepsilon_0\varepsilon_{zz}\ i\sigma_{zz}}\omega\frac{\mu_0\mu_{xz}\mu_{yz}}{\mu_{zz}}\right)H_x\\
&+i\left(\frac{k_x^2}{\omega\varepsilon_0\varepsilon_{zz}\ i\sigma_{zz}}-\omega\mu_0\mu_{yy}+\omega\mu_0\frac{\mu_{yz}^2}{\mu_z}\right)H_y+i\left(k_y\frac{\mu_{yz}}{\mu_{zz}}+k_x\frac{\omega\varepsilon_0\varepsilon_{xz}-i\sigma_{xz}}{\omega\varepsilon_0\varepsilon_{zz}-i\sigma_{zz}}\right)E_x
\end{aligned}
\tag{10}
$$

Let's write (10) in the matrix form:

$$\frac{d\vec{W}}{dz}=\hat{B}\vec{W}, \tag{11}$$

where $\vec{W}=(E_y,\,H_x,\,H_y,\,E_x)^t$ is column vector and $\hat{B}$ is generally a continuous matrix function in some interval (z_1,z_2) of variation of the argument z, hereinafter referred to as the matrix of coefficients.

$$\hat{B}=\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14}\\ b_{21} & b_{11} & b_{23} & b_{24}\\ -b_{24} & -b_{14} & b_{33} & b_{34}\\ -b_{23} & -b_{13} & b_{43} & b_{33}\end{bmatrix} \tag{12}$$

$$
\begin{aligned}
&b_{11}=ik_y\frac{\varepsilon_{yz}}{\varepsilon_z};\ b_{12}=i\left(\omega\mu_0\mu_x-\frac{k^2{}_y}{\omega\varepsilon_0\varepsilon_z}\right);\ b_{13}=i\frac{k_xk_y}{\omega\varepsilon_0\varepsilon_z};\ b_{14}=ik_y\frac{\varepsilon_{xz}}{\varepsilon_z}\\
&b_{21}=i\left(\omega\varepsilon_0\varepsilon_y-\frac{k^2{}_x}{\omega\mu_0\mu}-\frac{\omega\varepsilon_0\varepsilon_{yz}^2}{\varepsilon_z}\right);\ b_{22}=b_{11};\ b_{23}=-ik_x\frac{\varepsilon_{yz}}{\varepsilon_z};\ b_{24}\\
&\quad=i\left(\omega\varepsilon_0\varepsilon_{xy}+\frac{k_xk_y}{\omega\mu_0\mu}-\frac{\omega\varepsilon_0\varepsilon_{yz}\varepsilon_{xz}}{\varepsilon_z}\right);\\
&b_{31}=-b_{24};\ b_{32}=-b_{14};\ b_{33}=ik_x\frac{\varepsilon_{xz}}{\varepsilon_z};\ b_{34}=-i\left(\omega\varepsilon_0\varepsilon_x-\frac{k_y^2}{\omega\mu_0\mu}-\frac{\omega\varepsilon_0\varepsilon_{xz}^2}{\varepsilon_z}\right);\\
&b_{41}=-b_{23};\ b_{42}=-b_{13};\ b_{43}=i\left(\frac{k_x^2}{\omega\varepsilon_0\varepsilon_z}-\omega\mu_0\mu\right);\ b_{44}=b_{33}.
\end{aligned}
\tag{12.1}
$$

Thus, the system of equations (1)–(5) is reduced to a system of four ordinary differential equations of the first order (10) or to a matrix equation as given in (11).

4. Matricant structure

It is known that the normalized matrix of fundamental solutions of the system of equation (11) is called the matricant [12]. In our case, a matriciant is a normalized solution of a matrix equation $\frac{d\vec{W}}{dz} = \hat{B}\vec{W}$, $\vec{W} = (E_y, H_x, H_y, E_x)^t$, i.e. a solution that turns into a unit matrix at z = z0, where z0 is a fixed number in the interval (z1,z2), it is constructed by the method of successive approximations, where $\hat{B}$ denotes coefficient matrices. Any solution that has the meaning of a matrix of fundamental solutions has the form: $\hat{X} = \hat{T} \cdot \hat{C}$, $\hat{T} = \hat{T}(z, z_0)$ where $\hat{T}$ is matricant and $\hat{C}$ is an arbitrary constant matrix.

By the method of successive approximations based on recurrence relations:

$$\frac{d\hat{T}_k}{dz} = B\hat{T}_{k-1} \tag{13}$$

we obtain the matricant in the form of an infinite matrix exponential series [20]:

$$\hat{T} = \hat{E} + \int_0^z \hat{B}dz_1 + \int_0^z \int_0^{z_1} \hat{B}(z_1)\hat{B}(z_2)dz_1dz_2 + \ldots \tag{14}$$

A similar recurrence relation is also valid for constructing the inverse matricant:

$$\frac{d\hat{T}^{-1}{}_k}{dz} = -\hat{T}^{-1}{}_{k-1}\hat{B} \tag{15}$$

The element-wise comparison of each member of the rows:

$$\hat{T} = \sum_{n=1}^{\infty} \hat{T}_n, \quad \hat{T}^{-1} = \sum_{n=1}^{\infty} \hat{T}_n^{-1}, \quad \hat{T}_0 = \hat{E}, \quad \hat{T}_0^{-1} = \hat{E} \tag{16}$$

allows us to establish the structure $\hat{T}^{-1}$:

$$\hat{T}^{-1} = \begin{bmatrix} t_{22} & t_{12} & -t_{42} & -t_{32} \\ t_{21} & t_{11} & -t_{41} & -t_{31} \\ -t_{24} & -t_{14} & t_{44} & t_{34} \\ -t_{23} & -t_{13} & t_{43} & t_{33} \end{bmatrix}_{Even} - \begin{bmatrix} t_{22} & t_{12} & -t_{42} & -t_{32} \\ t_{21} & t_{11} & -t_{41} & -t_{31} \\ -t_{24} & -t_{14} & t_{44} & t_{34} \\ -t_{23} & -t_{13} & t_{43} & t_{33} \end{bmatrix}_{Odd} \tag{17}$$

in which t_{ij} is elements of the direct matricant $\hat{T}$, and $\hat{E}$ is the identity matrix.

$$\hat{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}_{Even} + \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix}_{Odd} \tag{18}$$

They mean that the even and odd elements of the matrix series were compared

$$\hat{T} = \hat{E} + \int_0^z \hat{B}dz_1 + \frac{1}{2!}\int_0^z \int_0^{z_1} \hat{B}(z_1)\hat{B}(z_2)dz_1dz_2 + \ldots$$

$$\hat{T}^{-1} = \hat{E} - \int_0^z \hat{B}dz_1 + \frac{1}{2!}\int_0^z \int_0^{z_1} \hat{B}(z_2)\hat{B}(z_1)dz_1dz_2 - \ldots$$

Each of these series is the sum of the matrices $\hat{T} = \sum_{n=0}^{\infty} \hat{T}_{(n)},$.

The index n is the same as the number of matrices multiplied under the integral sign $\hat{B}(z_i)$. There is a pattern; the members of the series with even and odd values are distinguished by n:

$$\hat{T}_{Even} = \sum_{n=0}^{\infty} \hat{T}_{(2n)}, \; \hat{T}_{Odd} = \sum_{n=0}^{\infty} \hat{T}_{(2n+1)}, \; \hat{T}_{Even}^{-1} = \sum_{n=0}^{\infty} \hat{T}_{(2n)}^{-1}, \; \hat{T}_{Odd}^{-1} = \sum_{n=0}^{\infty} \hat{T}_{(2n+1)}^{-1}$$

and the representation is given below:

$$\hat{T} = \hat{T}_{Even} + \hat{T}_{Odd}, \; \hat{T} = \hat{T}_{Even}^{-1} + \hat{T}_{Odd}^{-1}$$

The structure of the matricant as given in equations (17) and (18) represents the fundamental properties of solutions for equation (11).

Moreover, the Identities are given as under:

$$\hat{T}\,\hat{T}^{-1} = \hat{T}^{-1}\,\hat{T} = \hat{E}$$

and the invariant relations takes the following from:

$$\begin{array}{ll} t_{11}t_{22} + t_{12}t_{21} - t_{13}t_{24} - t_{14}t_{23} = 1; & -t_{21}t_{42} - t_{22}t_{41} + t_{23}t_{44} + t_{24}t_{43} = 0; \\ t_{11}t_{12} = t_{13}t_{14}; & -t_{21}t_{32} - t_{22}t_{31} + t_{23}t_{34} + t_{24}t_{33} = 0; \\ -t_{11}t_{42} - t_{12}t_{41} + t_{13}t_{44} + t_{14}t_{43} = 0; & -t_{31}t_{42} - t_{32}t_{41} + t_{33}t_{44} + t_{34}t_{43} = 1; \\ -t_{11}t_{32} - t_{12}t_{31} + t_{13}t_{34} + t_{14}t_{33} = 0; & t_{33}t_{34} = t_{32}t_{31}; \\ t_{22}t_{21} = t_{24}t_{23}; & t_{42}t_{41} = t_{44}t_{43}. \end{array} \tag{19}$$

5. Dispersion equations

Let us consider the applications of the obtained structures of matricants in the case of an unbounded periodically inhomogeneous medium. One of the main characteristics of such media are dispersion equations. By Knowing the structure of the matricants one can obtain a modified form for determining the roots of the monodromy matrix. This will make it possible to halve the degree of the characteristic equation, also it will allow to obtain roots, expressions for which are characterized by symmetry. The roots of the characteristic equation determine the dispersion law.

The Bloch's theorem implies that in case of the translational symmetry [27]:

$$\vec{W}(h) = e^{i\tilde{k}h}\vec{W}(0)$$

Using the monodromy matricant, we obtain:

$$\vec{W}(h) = \hat{T}\vec{W}(0)$$

Taking into account the last two ratios, we get:

$$(\hat{T} - e^{i\tilde{k}h}\hat{E})\vec{W}(0) = 0 \tag{20}$$

Multiplying by $-\hat{T}^{-1}e^{-i\tilde{k}h}$, we can get the following expression:

$$(\hat{\mathrm{T}}^{-1} - e^{-i\tilde{k}h}\hat{E})\vec{W}(0) = 0 \tag{21}$$

By setting the determinant of the matricant $\hat{T} - e^{i\tilde{k}h}\hat{E}$ in expression (20) to zero, gives the same dispersion law as in equation (21).

Combining equations (20) and (21) leads to a new modified form of the condition for the existence of nontrivial solutions of the matrix equation:

$$(\hat{T} + \hat{T}^{-1} - (e^{i\tilde{k}h} + e^{-i\tilde{k}h})\hat{E}) \cdot \vec{W}(0) = 0 \tag{22}$$

Introducing the matrix:

$$\hat{p} = \frac{1}{2}(\hat{T} + \hat{T}^{-1}) \tag{23}$$

from (22) we get:

$$\det(\hat{p} - \lambda\hat{E}) = 0 \tag{24}$$

The dispersion equations are solutions of the characteristic equation for the $\hat{p}$ matrix (24), that is, the dispersion equations in the general case have the form (25):

$$\cos\tilde{k}_i h = \tilde{p}_i = \lambda_i \tag{25}$$

In the case of general matricants (17), (18), the $\hat{p}$ matrix, based on the construction (23), takes the form:

$$\hat{p} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{11} & p_{23} & p_{24} \\ -p_{24} & -p_{14} & p_{33} & p_{34} \\ -p_{23} & -p_{13} & p_{43} & p_{33} \end{bmatrix}_{even} + \begin{bmatrix} q_{11} & 0 & q_{13} & q_{14} \\ 0 & -q_{11} & q_{23} & q_{24} \\ q_{24} & q_{14} & q_{33} & 0 \\ q_{23} & q_{13} & 0 & -q_{33} \end{bmatrix}_{odd} \tag{26}$$

in which q_{ij} are matrix elements of the form: $\hat{q} = \frac{1}{2}(\hat{T} - \hat{T}^{-1})$.

Replacing $\cos\tilde{k}h = \lambda$, solving the determinant in equation (24) and taking into account equation (26), we obtain

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \tag{27}$$

Where the roots are of the form:

$$\lambda_1 = -\frac{a}{4} - \frac{1}{2}A - \frac{1}{2}\sqrt{B - \frac{\gamma}{4A}},\ \lambda_2 = -\frac{a}{4} - \frac{1}{2}A + \frac{1}{2}\sqrt{B - \frac{\gamma}{4A}},$$
$$\lambda_3 = -\frac{a}{4} + \frac{1}{2}A - \frac{1}{2}\sqrt{B + \frac{\gamma}{4A}},\ \lambda_4 = -\frac{a}{4} + \frac{1}{2}A + \frac{1}{2}\sqrt{B + \frac{\gamma}{4A}},$$

in which $A = \sqrt{\frac{\sqrt[3]{\alpha + \sqrt{\alpha^2 - 4\beta^3}}}{3\sqrt[3]{2}} + \frac{a^2}{4} - \frac{2b}{3} + \frac{\sqrt[3]{2}\beta}{3\sqrt[3]{\alpha + \sqrt{\alpha^2 - 4\beta^3}}}}$,

$$B = -\frac{\sqrt[3]{\alpha + \sqrt{\alpha^2 - 4\beta^3}}}{3\sqrt[3]{2}} + \frac{a^2}{2} - \frac{4b}{3} - \frac{\sqrt[3]{2}\beta}{3\sqrt[3]{\alpha + \sqrt{\alpha^2 - 4\beta^3}}},$$

$$\alpha = 2b^3 - 9abc - 72db + 27c^2 + 27a^2d,\ \beta = b^2 - 3ac + 12d,\ \gamma = -a^3 + 4ab - 8c,$$
$$a = -2p_{11} - 2p_{33};$$
$$b = p_{11}^2 + 4p_{33}p_{11} + p_{33}^2 - q_{11}^2 - q_{33}^2 - p_{12}p_{21} + 2p_{14}p_{23} + 2p_{13}p_{24} - p_{34}p_{43} - 2q_{14}q_{23} - 2q_{13}q_{24};$$
$$\begin{aligned} c = &-2(p_{33}p_{11}^2 + (p_{33}^2 - q_{33}^2 + p_{14}p_{23} - p_{34}p_{43} - q_{14}q_{23} - q_{13}q_{24})p_{11} - p_{33}q_{11}^2 - p_{12}p_{23}p_{24} \\ &- p_{12}p_{21}p_{33} + p_{14}p_{23}p_{33} - p_{14}p_{24}p_{43} - p_{24}q_{11}q_{13} - q_{14}p_{23}q_{11} + p_{21}q_{13}q_{14} + p_{14}q_{11}q_{23} \\ &+ p_{34}q_{13}q_{23} - p_{33}q_{14}q_{23} - p_{33}q_{13}q_{24} + p_{43}q_{14}q_{24} + p_{12}q_{23}q_{24} - p_{24}q_{13}q_{33} + p_{23}q_{14}q_{33} \\ &- p_{14}q_{23}q_{33} + p_{13}(-p_{14}p_{21} + p_{11}p_{24} + p_{24}p_{33} - p_{23}p_{34} + q_{11}q_{24} + q_{24}q_{33}));\end{aligned}$$
$$\begin{aligned} d = &\, p_{11}^2p_{33}^2 - p_{11}^2p_{43}p_{34} + 2p_{34}p_{11}q_{13}q_{23} - 2p_{33}p_{11}q_{14}q_{23} - 2p_{33}p_{11}q_{13}q_{24} \\ &+ 2p_{43}p_{11}q_{14}q_{24} - p_{12}p_{21}p_{33}^2 - p_{33}^2q_{11}^2 + p_{34}p_{43}q_{11}^2 - p_{24}^2q_{13}^2 - p_{21}p_{34}q_{13}^2 - p_{23}^2q_{14}^2 \\ &- p_{21}p_{43}q_{14}^2 + q_{14}^2q_{23}^2 - p_{12}p_{34}q_{23}^2 + q_{13}^2q_{24}^2 - p_{12}p_{43}q_{24}^2 - (p_{11}^2 - q_{11}^2 - p_{12}p_{21})q_{33}^2 \\ &- 2p_{12}p_{23}p_{24}p_{33} + p_{23}^2p_{12}p_{34} + p_{24}^2p_{12}p_{43} + p_{12}p_{21}p_{34}p_{43} - 2p_{24}p_{33}q_{13}q_{11} \\ &- 2p_{23}p_{34}q_{13}q_{11} - 2p_{23}p_{33}q_{14}q_{11} + 2p_{24}p_{43}q_{14}q_{11} + 2p_{24}p_{23}q_{13}q_{14} + 2p_{21}p_{33}q_{13}q_{14} \\ &+ p_{14}^2(p_{23}^2 - q_{23}^2 + p_{21}p_{43}) + 2p_{12}p_{33}q_{23}q_{24} - 2q_{13}q_{14}q_{23}q_{24} + p_{13}^2(p_{24}^2 - q_{24}^2 + p_{21}p_{34}) \\ &- 2(p_{11}(p_{24}q_{13} - p_{23}q_{14}) - (p_{12}p_{24} + q_{11}q_{14})q_{23} + (p_{12}p_{23} + q_{11}q_{13})q_{24})q_{33} \\ &- 2p_{14}(-p_{33}q_{11}q_{23} - p_{24}q_{13}q_{23} + p_{43}q_{11}q_{24} + p_{23}q_{13}q_{24} + p_{13}(p_{23}p_{24} + p_{21}p_{33} - q_{23}q_{24}) \\ &+ (p_{23}q_{11} - p_{21}q_{13})q_{33} + p_{11}(-p_{23}p_{33} + p_{24}p_{43} + q_{23}q_{33})) \\ &+ 2p_{13}(-p_{34}q_{11}q_{23} - p_{24}q_{14}q_{23} + p_{33}q_{11}q_{24} + p_{23}q_{14}q_{24} + (p_{24}q_{11} - p_{21}q_{14})q_{33} \\ &+ p_{11}(p_{24}p_{33} - p_{23}p_{34} + q_{24}q_{33}));\end{aligned}$$

Thus, the dispersion equations in the general case for a periodically inhomogeneous medium take the form:

$$\cos\tilde{k}_1h = -\frac{a}{4} - \frac{1}{2}A - \frac{1}{2}\sqrt{B - \frac{\gamma}{4A}},\ \cos\tilde{k}_2h = -\frac{a}{4} - \frac{1}{2}A + \frac{1}{2}\sqrt{B - \frac{\gamma}{4A}},$$
$$\cos\tilde{k}_3h = -\frac{a}{4} + \frac{1}{2}A - \frac{1}{2}\sqrt{B + \frac{\gamma}{4A}},\ \cos\tilde{k}_4h = -\frac{a}{4} + \frac{1}{2}A + \frac{1}{2}\sqrt{B + \frac{\gamma}{4A}}. \quad (28)$$

where, h shows the distance at which a periodically inhomogeneous medium can be considered homogeneous.

6. Matricant of a periodically inhomogeneous layer

Let us consider the finite periodic structures and analytic representation $\hat{T}^n$.

If a periodically inhomogeneous layer of thickness H has n periods, i.e. H = nh, then the calculation of the matricant leads to the calculation of the monodromy matrix to the n degree:

$$\hat{T}(H) = \hat{T}^n(h)$$

The introduction of the $\hat{p}$ matrix according to (23) based on the known structure $\hat{T}^{-1}$ has made it possible to obtain the analytical representation $\hat{T}(H)$ in terms of the monodromy matrix $\hat{T}(h)$.

Directly from (23) we obtain:

$$\hat{T}^2 = 2\hat{p}\hat{T} - \hat{E} \quad (29)$$

Reapplying to (n-1) and nth periodic layer, multiplying (29) on the right by $\hat{T}$:

$$\hat{T}^3 = (4\hat{p}^2 - \hat{E})\hat{T} - 2\hat{p},\ \hat{T}^4 = (8\hat{p}^3 - 4\hat{p})\hat{T} - (4\hat{p}^2 - \hat{E})\ldots$$

leads to recurrence relations:

$$\hat{T}^n = P_n(\hat{p})\hat{T} - P_{n-1}(\hat{p}) \tag{30}$$

This is the analytical view $\hat{T}^n$.

The $\hat{p}$ structure allows matrix polynomials $P_n(\hat{p})$ to be reduced to algebraic Chebyshev—Gegenbauer polynomials of the second kind and represented in the $\hat{T}^n$ form:

$$\hat{T}^n = \sum_{i=1}^{4} \hat{P}_i(P_n(\tilde{p}_i)\hat{T} - P_{n-1}(\tilde{p}_i)\hat{E}) \tag{31}$$

7. Matricant of the averaged medium

We are averaging the medium under the condition $\lambda \gg h$ (in which λ is the wavelength, h is the period of inhomogeneity). Whence from the fact that $\tilde{k}h = \frac{2\pi h}{\lambda} \ll 1$ we obtain the expansion of the dispersion equation (26) in the form

$$\tilde{p}_i = \cos \tilde{k}_i h \cong 1 - \frac{\tilde{k}_i^2 h^2}{2} \quad \text{or} \quad \sqrt{1 - \tilde{p}_i^2} \approx \tilde{k}_i h = \frac{\tilde{k}_i H}{n}, \tag{32}$$

in which $H = nh$ is the total layer thickness, and n is the number of periods in a layer.

$$\hat{P}_{(2)} = \hat{E} + \frac{\langle \hat{B} \rangle^2 h^2}{2}; \ \langle \hat{B} \rangle = \frac{1}{h}\int_0^h \hat{B}dz \tag{33}$$

Further approximation [16] allows us to introduce the relation $\hat{T} - \tilde{p}_i\hat{E} \cong \langle \hat{B} \rangle h$. Wherein:

$$P(\tilde{p}_i)\hat{T} - P_{n-1}(\tilde{p}_i)\hat{E} = \cos \tilde{k}_i H + \frac{\langle \hat{B} \rangle}{\tilde{k}_i} \sin \tilde{k}_i H \tag{34}$$

Substituting (34) into (31), we get:

$$\hat{T}_{ycp} = \sum_{i=1}^{4} P_i\left[\hat{E} \cos \tilde{k}_i H + \frac{\langle \hat{B} \rangle}{\tilde{k}_i} \sin \tilde{k}_i H\right], \tag{35}$$

in which

$$P_i = \frac{(\hat{P}_{(2)} - \tilde{p}_j)(\hat{P}_{(2)} - \tilde{p}_k)(\hat{P}_{(2)} - \tilde{p}_l)}{(\tilde{p}_i - \tilde{p}_j)(\tilde{p}_i - \tilde{p}_k)(\tilde{p}_i - \tilde{p}_l)}; \ (\mathrm{i, j, k, l} = 1, 2, 3, 4; \ i \neq j \neq k \neq l) \tag{36}$$

In formulas (36) $\tilde{p}_i$ are the roots of the characteristic equation following from the condition:

$$\det(\hat{p}_{(2)} - \lambda\hat{E}) = 0 \tag{37}$$

Along with the construction of the matricant (37), knowledge of the roots makes it possible to obtain the equations of the indicatrices of electromagnetic waves of different polarization. The indicatrix curves are determined by the equations:

$$1 - \frac{\tilde{k}_i^2 h^2}{2} = \tilde{p}_i^{(2)} \Rightarrow \tilde{k}_i^2 = \frac{2(1 - \tilde{p}_i^{(2)})}{h^2} \tag{38}$$

8. Equations of indicatrices

An indicatrice is a line or surface that visually characterizes some property of the object under investigation, for example, an indicatrice in optics [28].

The equations of indicatrices for an unbounded periodically inhomogeneous anisotropic structure follow from the low-frequency averaging of the dispersion equation (25). To begin with, we transform equations (32), (33) and (37) to a convenient form:

$$\det(\hat{E} + B^2h^2/2 - \hat{E} + \hat{E}\ (k_i h)^2/2) = 0 = >\det(B^2 + \hat{E}\ k_i{}^2) = 0 \tag{39}$$

Let us solve the differential matrix equation (11), taking into account (12), (12.1) for crystals described by the dielectric permeability, magnetic permeability and conductivity of the form:

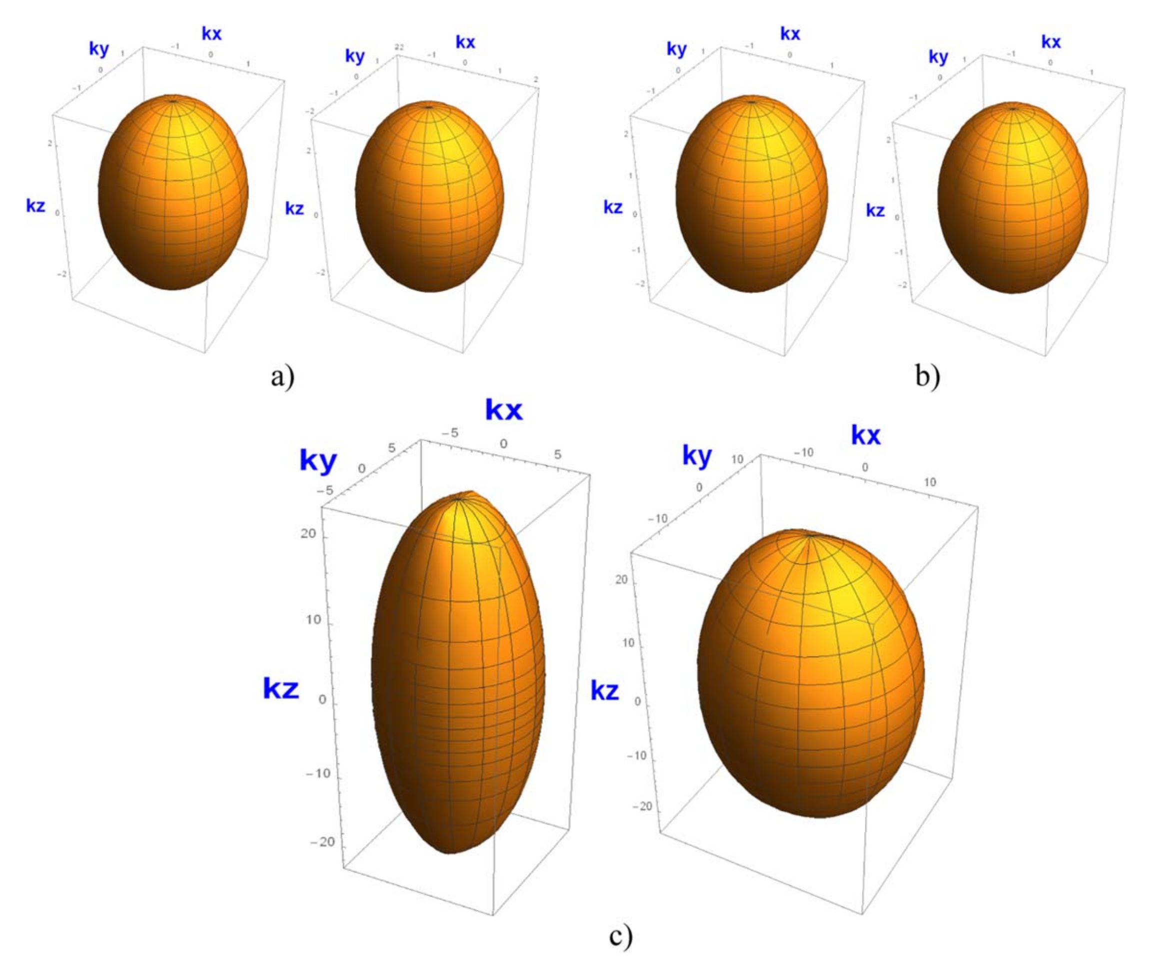

Figure 1. Wave vector indicatrices: (a) HIO_3, (b) $Ba_2NaNb_5O_{15}$, (c) $LiIO_3$.

$$\hat{\varepsilon} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & \varepsilon_{yz} \\ 0 & \varepsilon_{yx} & \varepsilon_z \end{bmatrix},\ \mu_{ij} = const,\ \sigma_{ij} = 0.$$

We obtain the equation of the wave vector indicatrices for the specified type of media:

$$\begin{aligned} &\frac{1}{\varepsilon_z}((k^2 + k_x{}^2 + k_y{}^2)(k_x{}^2\varepsilon_x + 2k_x k_y \varepsilon_{xy} + k_y{}^2\varepsilon_y + k^2\varepsilon_z) \\ &+ \varepsilon_0\mu_0\mu\ \omega^2((k_x{}^2 + k_y{}^2)(\varepsilon_{xy}{}^2 - \varepsilon_x\varepsilon_y) - ((k^2 + k_x{}^2)\varepsilon_x + 2k_x k_y \varepsilon_{xy} \\ &+ (k^2 + k_y{}^2)\varepsilon_y)\ \varepsilon_z + \varepsilon_0{}^2\mu_0{}^2\mu^2\ \omega^4(-\varepsilon_{xy}{}^2 + \varepsilon_x\varepsilon_y)) = 0 \end{aligned} \tag{40}$$

Let's introduce the spherical coordinate system:

$$\begin{cases} \mathrm{k_x} = \mathrm{k}\sin\varphi\ \cos\theta, \\ \mathrm{k_y} = \mathrm{k}\sin\varphi\ \sin\theta, \\ \mathrm{k_z} = \mathrm{k}\cos\varphi. \end{cases}$$

Let's set the parameters of electromagnetic waves, adequate to the long-wave approximation: $\omega^2\varepsilon_0\mu_0 = \omega^2/\mathrm{c}^2 \approx \omega^2/10^{17} = 1\ \text{м}^{-2}$, meter and decimeter radio waves.

The environment parameters [29, 30]:

(1) HIO_3 $\hat{\varepsilon} = \{7, 2; 8, 0; 6, 9\}$, $\varepsilon_{xy} = 0$, $\mu = 1$ (figure 1(a));

(2) $Ba_2NaNb_5O_{15}$ $\hat{\varepsilon} = \{5, 63; 5, 62; 6, 1\}$, $\varepsilon_{xy} = 0$, $\mu = 1$ (figures 1(b) and 2);

(3) $LiIO_3$ for $\hat{\varepsilon} = \{500; 554; 65\}$, $\varepsilon_{xy} = 0$, $\mu = 1$, instead of $\varepsilon_{xx} = 554$, $\varepsilon_{xx} = 500$ is taken, otherwise a flat curve is obtained (figure 1(c));

(4) $LiNbO_3$ $\hat{\varepsilon} = \{78; 78; 32\}$, $\varepsilon_{xy} = 0$, $\mu = 1$ (figure 3(a));

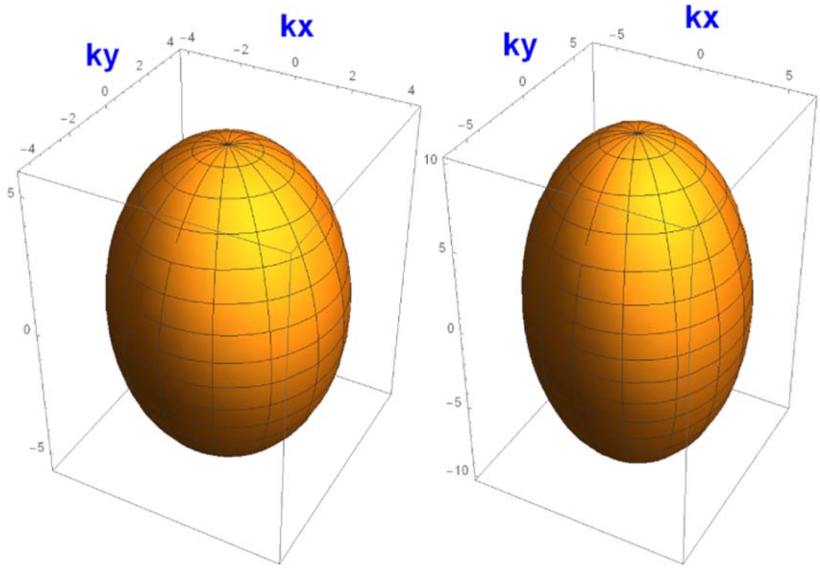

Figure 2. The indicatrix of the wave vector in a hypothetical medium with $\varepsilon_{xx} = 80$, $\varepsilon_{yy} = 50$, $\varepsilon_{zz} = 60$, $\varepsilon_{xy} = 30$.

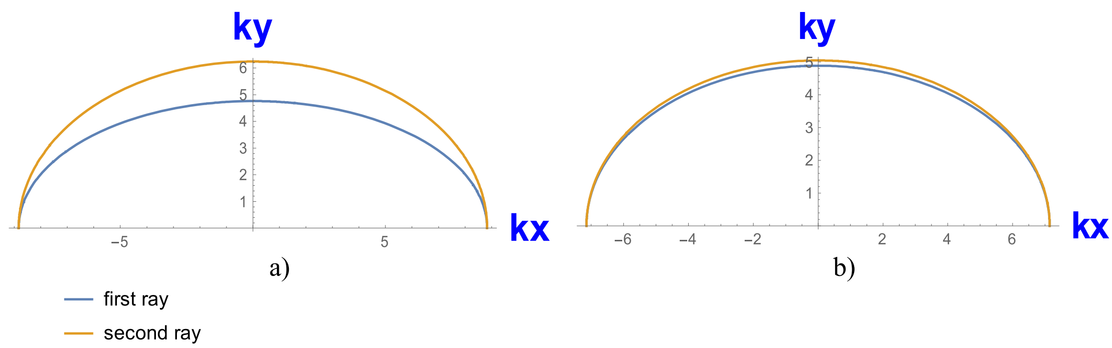

Figure 3. Curve of the wave vector indicatrix for the crystal: (a)–$LiNbO_3$, (b)– $LiTaO_3$.

(5) $LiTaO_{3,}$ $\hat{\varepsilon} = \{51;\ 51;\ 45\}$, $\varepsilon_{xy} = 0$, $\mu = 1$ (figure 3(b)).

We obtain the following non-negative values of the wave vectors of both rays:

Thus, figure 1 shows the indicatrices of wave vectors with different parameters of the medium for a visual demonstration of the use of waves. For a complete picture of indicatrices, let us consider a hypothetical environment where the parameters are equal to $\varepsilon_{xx} = 80$, $\varepsilon_{yy} = 50$, $\varepsilon_{zz} = 60$, $\varepsilon_{xy} = 30$. Figure 2 shows the indicatrix of the wave vector in a hypothetical medium. Each figure shows in pairs the indicatrices for the ordinary and extraordinary rays.

As seen from the figure, the shape of the indicatrices does not change due to the presence of a small parameter . However, for very large values compared to the rest, complex roots appear and the surface is not displayed.

For a lithium niobate crystal $LiNbO_3$ at frequencies below the intrinsic acoustic resonance $\hat{\varepsilon} = \{78;\ 78;\ 32\}$ the wave vector, in view of the tetragonal system, turned out as shown in figure 3(a). Similarly to lithium tantalate $LiTaO_3$ $\hat{\varepsilon} = \{51;\ 51;\ 45\}$ are also shown in figure 3(b).

The results obtained describe the wave processes graphically, indicating the distribution of the wave vector along the directions of the main optical axis in crystals of low symmetry.

9. Conclusion

Thus, as a result of theoretical studies, a system of ordinary differential equations with variable coefficients was obtained, the solution of which determines free electromagnetic fields in one-dimensionally inhomogeneous anisotropic media with conductivity. The structure of the matrix of coefficients for electromagnetic waves in anisotropic media is determined. For anisotropic media with conductivity inhomogeneous along the Oz axis, the structure of the direct and inverse matrices of Maxwell's equations is constructed. As has been shown, using the matrix method, one can obtain dispersion equations for a periodically inhomogeneous anisotropic medium. And also the matricant of the averaged anisotropic medium with conductivity is obtained in the long-wave approximation in an explicit analytical form.

Mathematical modeling shows that it is possible to obtain indicatrix curves for some transparent non-magnetic crystals. The matricant method allows one to model the wave vector in lower systems.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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